

Scattering of a scalar relativistic particle by the hyperbolic tangent potential

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Hypergeometric functions, Klein-Gordon equation, Scattering theory

Abstract

We solve the Klein-Gordon equation in the presence of the hyperbolic tangent potential. The scattering solutions are derived in terms of hypergeometric functions. The reflection R and transmission T coefficients are calculated in terms of Gamma function and, superradiance is discussed, when the reflection coefficient R is greater than one.

1 Introduction

The study of the solution of the Klein-Gordon equation with different potentials has been extensively studied in recent years, for bound states and scattering solutions [1, 2, 3, 4, 5, 6, 7]. The scattering solutions to the Dirac equation also have been studied for several potentials, a particular case is the Hulthén potential [8]. The superradiance phenomenon, when the reflection coefficient R is greater than one, has been widely discussed. Manogue [9] discussed the superradiance on a potential barrier for Dirac and Klein-Gordon equations. Sauter [10] and Cheng [11] have studied the same phenomenon for the hyperbolic tangent potential with the Dirac equation. Superradiance for the Klein-Gordon equation with this particular potential has been studied for Cheng too [11]. The hyperbolic tangent potential even wake interest, recently has discussed the bound states of scalar particle in presence of the truncated hyperbolic tangent potential [12] and the hyperbolic tangent potential [13].

In this paper we have calculated the scattering solutions of the Klein-Gordon equation in terms of hypergeometric functions in presence of the hyperbolic tangent potential. The reflection R and transmission T coefficients are calculated in terms of Gamma function. The behaviour of the reflection R and transmission T coefficients is studied for five different regions of energy. We have observed for some region that $R > 1$ and $T < 0$, so the phenomenon of superradiance is observed in this potential [11, 14].

This paper is organized of the following way. Section 2 shows the one-dimensional Klein-Gordon equation. In section 3 the hyperbolic tangent potential is shown. Section 4 shows the scattering solutions and the behaviour of the reflection R and transmission T coefficients. Finally, in section 5 conclusions are discussed.

2 The Klein-Gordon equation

The one-dimensional Klein-Gordon equation to solve is, in natural units $\hbar = c = 1$ [15]

$$\frac{d^2\phi(x)}{dx^2} + \{[E - V(x)]^2 - m^2\} \phi(x) = 0, \quad (1)$$

where E is the energy, $V(x)$ is the potential and, m is the mass of the particle.

3 The hyperbolic tangent potential

The hyperbolic tangent potential is defined as

$$V(x) = a \tanh(bx), \quad (2)$$

where a represents the height of the potential and b gives the smoothness of the curve. The form of the hyperbolic tangent potential is showed in the Fig. 1. From Fig. 1 we can note that the hyperbolic tangent potential reduces to a step potential for $b \rightarrow \infty$.

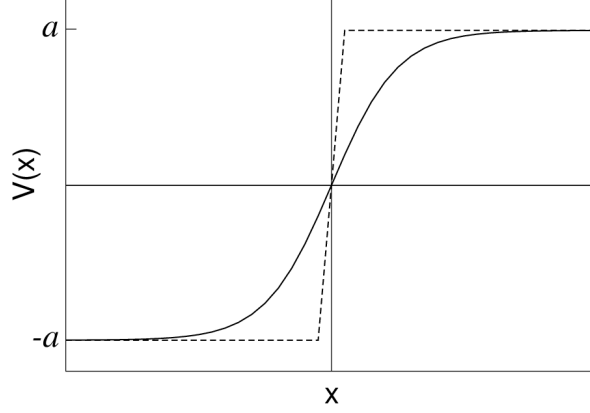


Figure 1: Hyperbolic tangent potential for $a = 5$ with $b = 2$ (solid line) and $b = 50$ (dotted line).

4 Scattering States

In order to consider the scattering solutions, we solve the differential equation

$$\frac{d^2\phi(x)}{dx^2} + \{[E - a \tanh(bx)]^2 - m^2\} \phi(x) = 0. \quad (3)$$

On making the substitution $y = -e^{2bx}$, Eq. (3) becomes

$$4b^2 y \frac{d}{dy} \left[y \frac{d\phi(y)}{dy} \right] + \left[\left(E + a \frac{1+y}{1-y} \right)^2 - m^2 \right] \phi(y) = 0. \quad (4)$$

Putting $\phi(y) = y^\alpha (1-y)^\beta f(y)$, Eq. (4) reduces to the hypergeometric differential equation

$$y(1-y)f'' + [(1+2\alpha) - (2\alpha+2\beta+1)y]f' - (\alpha+\beta-\gamma)(\alpha+\beta+\gamma)f = 0, \quad (5)$$

where the primes denote derivate with respect to y and the parameters α , β , and γ are

$$\alpha = i\nu \text{ with } \nu = \frac{\sqrt{(E+a)^2 - m^2}}{2b}, \quad (6)$$

$$\beta = \lambda \text{ with } \lambda = \frac{b + \sqrt{b^2 - 4a^2}}{2b}, \quad (7)$$

$$\gamma = i\mu \text{ with } \mu = \frac{\sqrt{(E-a)^2 - m^2}}{2b}. \quad (8)$$

Eq. (5) has the general solution in terms of Gauss hypergeometric functions ${}_2F_1(\mu, \nu, \lambda; y)$ [16]

$$\begin{aligned} \phi(y) = & C_1 y^\alpha (1-y)^\beta {}_2F_1(\alpha + \beta - \gamma, \alpha + \beta + \gamma, 1 + 2\alpha; y) \\ & + C_2 y^{-\alpha} (1-y)^\beta {}_2F_1(-\alpha + \beta - \gamma, -\alpha + \beta + \gamma, 1 - 2\alpha; y). \end{aligned} \quad (9)$$

In terms of variable x Eq. (9) becomes

$$\begin{aligned} \phi(x) = & c_1 (-e^{2bx})^{i\nu} (1 + e^{2bx})^\lambda {}_2F_1(i\nu + \lambda - i\mu, i\nu + \lambda + i\mu, 1 + 2i\nu; -e^{2bx}) \\ & + c_2 (-e^{2bx})^{-i\nu} (1 + e^{2bx})^\lambda {}_2F_1(-i\nu + \lambda + i\mu, -i\nu + \lambda - i\mu, 1 - 2i\nu; -e^{2bx}). \end{aligned} \quad (10)$$

From Eq. (10) the incident and reflected waves are

$$\phi_{\text{inc}}(y) = d_1 (1 + e^{2bx})^\lambda e^{2ib\nu x} {}_2F_1(i\nu + \lambda - i\mu, i\nu + \lambda + i\mu, 1 + 2i\nu; -e^{2bx}). \quad (11)$$

$$\phi_{\text{ref}}(y) = d_2 (1 + e^{2bx})^\lambda e^{-2ib\nu x} {}_2F_1(-i\nu + \lambda + i\mu, -i\nu + \lambda - i\mu, 1 - 2i\nu; -e^{2bx}). \quad (12)$$

Using the relation [16]

$$\begin{aligned} {}_2F_1(a, b, c; z) = & \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)} (-z)^{(-a)} {}_2F_1(a, 1-c+a, 1-b+a; z^{-1}) \\ & + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)} (-z)^{(-b)} {}_2F_1(b, 1-c+b, 1-a+b; z^{-1}). \end{aligned} \quad (13)$$

The transmitted wave becomes

$$\phi_{\text{trans}}(x) = d_3 e^{-2b\lambda x} (1 + e^{2bx})^\lambda e^{2ib\mu x} {}_2F_1(i\nu + \lambda - i\mu, -i\nu + \lambda - i\mu, 1 - 2i\mu; -e^{-2bx}). \quad (14)$$

As the incident wave is equal to the sum of the transmitted wave and the reflected wave

$$\phi_{\text{inc}}(x) = A \phi_{\text{trans}}(x) + B \phi_{\text{ref}}(x). \quad (15)$$

We used again the relation (13) and the equation for $\phi_{\text{trans}}(x)$ to find

$$\phi_{\text{inc}}(x) = A (1 + e^{2bx})^\lambda e^{2ib\nu x} {}_2F_1(i\nu + \lambda - i\mu, i\nu + \lambda + i\mu, 1 + 2i\nu; -e^{2bx}). \quad (16)$$

$$\phi_{\text{ref}}(x) = B (1 + e^{2bx})^\lambda e^{-2ib\nu x} {}_2F_1(-i\nu + \lambda + i\mu, -i\nu + \lambda - i\mu, 1 - 2i\nu; -e^{2bx}). \quad (17)$$

Where the coefficients A and B in Eqs. (16) and (17) can be expressed in terms of the Gamma function as

$$A = \frac{\Gamma(1 - 2i\mu)\Gamma(-2i\nu)}{\Gamma(-i\nu + \lambda - i\mu)\Gamma(1 - i\nu - \lambda - i\mu)}. \quad (18)$$

$$B = \frac{\Gamma(1 - 2i\mu)\Gamma(2i\nu)}{\Gamma(i\nu - \lambda - i\mu)\Gamma(1 + i\nu - \lambda - i\mu)}. \quad (19)$$

When $x \rightarrow \pm\infty$ the $V \rightarrow \pm a$ and the asymptotic behaviour of Eqs. (14), (16) and, (17) are plane waves with the relation of dispersion ν and μ ,

$$\phi_{\text{inc}}(x) = A e^{2ib\nu x}, \quad (20)$$

$$\phi_{\text{ref}}(x) = B e^{-2ib\nu x}, \quad (21)$$

$$\phi_{\text{trans}}(x) = e^{2ib\mu x}. \quad (22)$$

In order to find R and T , we used the definition of the electrical current density for the one-dimensional Klein-Gordon equation (1)

$$\vec{j} = \frac{i}{2} \left(\phi^* \vec{\nabla} \phi - \phi \vec{\nabla} \phi^* \right). \quad (23)$$

The current as $x \rightarrow -\infty$ can be decomposed as $j_L = j_{\text{inc}} - j_{\text{ref}}$ where j_{inc} is the incident current and j_{ref} is the reflected one. Analogously we have that, on the right side, as $x \rightarrow \infty$ the current is $j_R = j_{\text{trans}}$, where j_{trans} is the transmitted current [1].

The reflection coefficient R , and the transmission coefficient T , in terms of the incident j_{inc} , reflected j_{ref} , and transmission j_{trans} currents are

$$R = \frac{j_{\text{ref}}}{j_{\text{inc}}} = \frac{|B|^2}{|A|^2}. \quad (24)$$

$$T = \frac{j_{\text{trans}}}{j_{\text{inc}}} = \frac{\mu}{\nu} \frac{1}{|A|^2}. \quad (25)$$

The reflection coefficient R , and the transmission coefficient T satisfy the unitary relation $T + R = 1$ and are expressed in terms of the coefficients A and B , therefore are expressed in terms of the Gamma function and they are determined with the software Maple 18.

The dispersion relation ν and μ must be positive because it corresponds to an incident particle moved from left to right and, their sign depends on the group velocity, define by [17]

$$\frac{dE}{d\nu'} = \frac{\nu'}{E + a} \geq 0. \quad (26)$$

$$\frac{dE}{d\mu'} = \frac{\mu'}{E - a} \geq 0. \quad (27)$$

For the hyperbolic tangent potential we have five different regions, these regions are observed in table 1.

It is important to note that in the regions $a + m > E > a - m$ and $-a + m > E > -a - m$ the dispersion relations μ and ν are imaginary pure and the transmitted wave is attenuated, so $R = 1$. In the region $a - m > E > -a + m$, $\mu' < 0$ and, $\nu' > 0$ we have that $R > 1$, so superradiance occurs.

Figs. 2(a) and 2(b) show the reflection R and transmission T coefficients for $E > m$, $a = 5$ and $b = 2$. Figs. 3(a) and 3(b) show the reflection R and

$E > a + m$	$\nu' > 0$	$\nu \in \Re$	$\mu' > 0$	$\mu \in \Re$
$a + m > E > a - m$	$\nu' > 0$	$\nu \in \Re$		$\mu \in \Im$
$a - m > E > -a + m$	$\nu' > 0$	$\nu \in \Re$	$\mu' < 0$	$\mu \in \Re$
$-a + m > E > -a - m$		$\nu \in \Im$	$\mu' < 0$	$\mu \in \Re$
$E < -a - m$	$\nu' < 0$	$\nu \in \Re$	$\mu' < 0$	$\mu \in \Re$

Table 1: Regions for ν' and μ' .

transmission T coefficients R for $E > m$, $a = 5$ and $b = 50$. We observed in the figures that in the region $a - m > E > m$ the reflection coefficient R is bigger than one whereas the coefficient of transmission T becomes negative, so we observed superradiance [11, 14] and that the coefficients R and T satisfy the unitary condition $T + R = 1$. The hyperbolic tangent potential is useful to study the superradiance phenomenon.

5 Conclusion

In this paper we have discussed the scattering solutions of the Klein-Gordon equation in presence of the hyperbolic tangent potential. The solutions are determined in terms of hypergeometric function. The reflection R and transmission T coefficients are determined in terms of the Gamma function. We have shown that for the region where $a - m > E > m$, the phenomenon of superradiance occurs.

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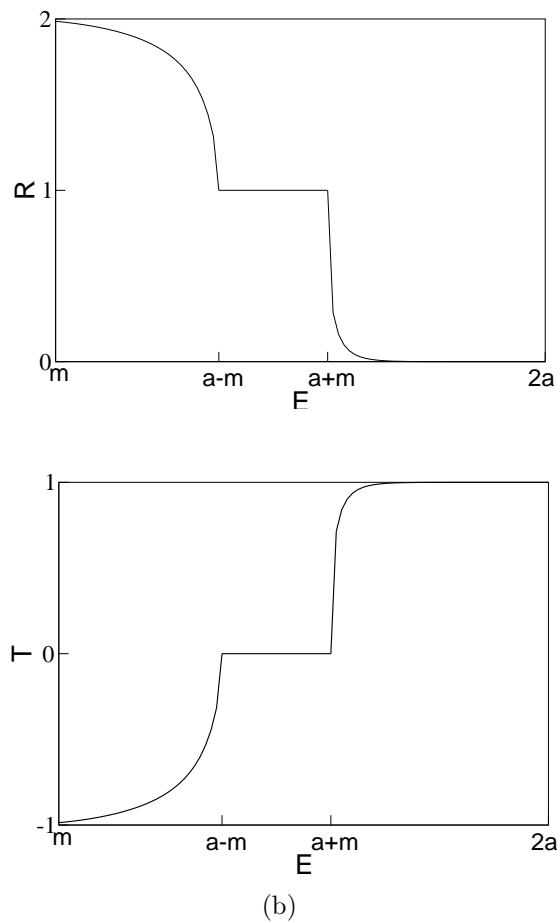


Figure 2: The reflection R and transmission T coefficients varying energy E for the relativistic hyperbolic tangent potential for $a = 5$, $b = 2$ and, $m = 1$.

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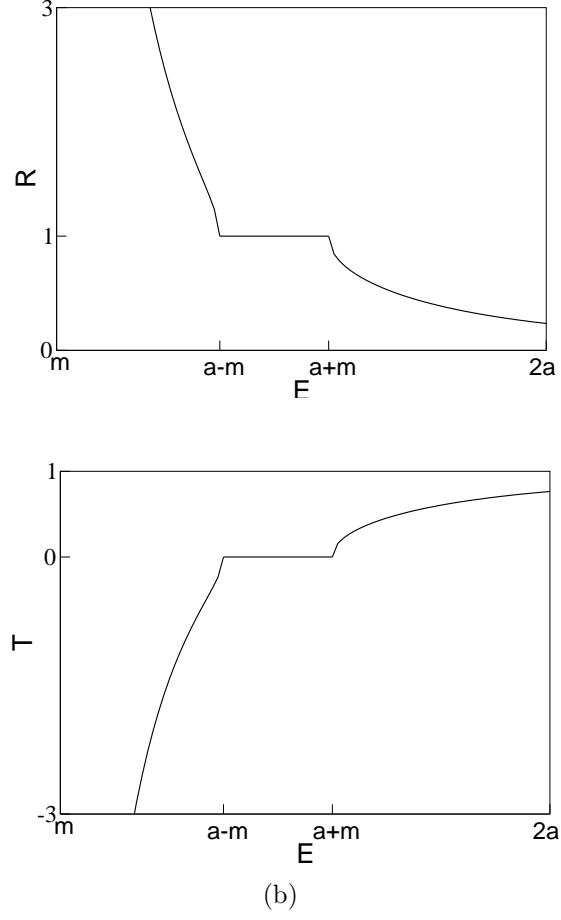


Figure 3: The reflection R and transmission T coefficients varying energy E for the relativistic hyperbolic tangent potential for $a = 5$, $b = 50$ and, $m = 1$.

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